Balancing Flexible Rotating Shafts with an Initial Bend

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The synchronous whirl of a rotating, flexible shaft induced by an initial bend is similar, though slightly different, than that induced by a pure mass unbalance. The differences are due to differences in the forcing effects produced by these phenomena. A discussion of the problems associated with balancing bent, flexible shafts is presented. Experimental results are reported for a long, flexible shaft which demonstrate the effects of initial bend at speeds up to and beyond the fourth critical speed. Included are the results of a series of very successful balancing tests.

Introduction

IT is well-known that a rotating, flexible shaft can experience a forced, synchronous whirl due to mass unbalance. A somewhat similar whirl can also be induced if the shaft has an initial bend, i.e., its central axis is bowed, when the shaft is not rotating. This initial bend is distinct from the sag which is produced in a horizontal shaft by its weight. Although the two forced whirls are similar, some significant differences in their character arise because in the case of mass unbalance the forcing is proportional to the square of the shaft speed, whereas the forcing effect produced by an initial bend does not vary with shaft speed. These differences can lead to complications and misunderstandings in balancing.

The first author drew attention to some of these differences 17 years ago and a few related publications have been produced since by other workers. The present paper has two principal aims. First, a further discussion of the problems associated with balancing bent, flexible shafts. This discussion is intended to remind practical balancing engineers of potential misunderstandings in balancing and, also, to present a revised balancing procedure.

The second purpose of this paper is to report some experimental results for a long, flexible shaft which demonstrate the effects of initial bend at speeds up to values in excess of the fourth critical speed. Balancing trials with this shaft are also described and the authors believe the paper to be the first, practical demonstration of the effects of initial bend at speeds up to the fourth critical speed.

Whirl Due to Unbalance and Initial Bend

For simplicity we will consider a vertical shaft with distributed mass and stiffness. In this way the effects of an initial bend can be examined without any confusion due to the deflection caused by shaft weight for a horizontal shaft. If a shaft is horizontal, of course, then within the confines of linear theory the deflection due to weight may be added to the displacement of a corresponding vertical shaft to determine the overall deflection.

A typical cross section of such a shaft is shown in Fig. 1, which also defines two systems of reference axes. The point O lies on the bearing axis. When deflected, in the plane of the cross section, axes OXY are stationary, whereas axes OUV rotate about the bearing axis with the steady shaft speed Ω. Thus, the displacement of the midpoint E of the cross section can be represented by complex coordinates ξ and η relative to the fixed and rotating axes, respectively. Thus,

$$\xi = x + iy \qquad \eta = u + iv \tag{1}$$

and

$$\xi = \eta e^{i\Omega t} \tag{2}$$

The location of points along the shaft is expressed by means of a coordinate, z, along the undeflected direction of the bearing axis. Thus, x, y, u, v, ξ, and η are functions of z. If the shaft has a conventional unbalance, then the mass center C of the thin slice in Fig. 1 does not coincide with the geometric center E. The eccentricity EC can be expressed in terms of its projections on the rotating axes OUV in the form,

$$\bar{a}(z) = a(z) + ia'(z) \tag{3}$$

The complex representation and the functional dependence on z indicate that the mass eccentricity probably varies in both magnitude and orientation around the shaft for differing cross sections along the shaft.

If the shaft has an initial bend, or bow, then even when the shaft is not rotating, point E does not coincide with the bearing axis through O. This initial displacement of the point E can be formulated mathematically as

$$\eta_0 = u_0(z) + iv_0(z) \tag{4}$$

Even at very low speeds the axis of the shaft is bowed away from the bearing axis, so that a transverse transducer located at a distance z along the shaft would detect the initial bow in the form

$$\xi_0 = \eta_0 e^{i\Omega t} \tag{5}$$

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The equations of motion of such a shaft can be expressed concisely in complex form as follows:

$$\rho A \frac{\partial^2 \xi}{\partial t^2} + (b+h)\frac{\partial \xi}{\partial t} - ih\Omega\xi + \frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2}{\partial z^2}(\xi - \xi_0)\right] = \Omega^2 \rho A \bar{a}(z)e^{i\Omega t} \tag{6}$$

where ρA is the mass per unit length of the shaft, EI the bending stiffness, and b, h represent external and internal (rotating) damping, respectively. The initial bow, therefore, generates a forcing effect at any speed Ω because, when the shaft is deflected by any amount ξ from the bearing axis, the restoring force is a function of $\xi - \xi_0$, according to Bernoulli-Euler beam theory and not just ξ, which applies to a shaft with no initial bend. The whirl of an initially bowed shaft can be likened to that of a shaft with no initial bend with an additional forcing term at a cross section of thickness δz of the form

$$\delta z \frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2 \xi_0}{\partial z^2}\right] \tag{7}$$

This interpretation is seen clearly if Eq. (6) is rewritten as

$$\rho A \frac{\partial^2 \xi}{\partial t^2} + (b+h)\frac{\partial \xi}{\partial t} - ih\Omega\xi + \frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2 \xi}{\partial z^2}\right] = \Omega^2 \rho A \bar{a}(z)e^{i\Omega t} + \frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2 \xi_0}{\partial z^2}\right] \tag{8}$$

A modal solution for the whirling motion has been developed and reported elsewhere,[1] so that only the results of the modal analysis will be recalled here. An infinite set of orthogonal, principal modes can be formulated in terms of characteristic functions $\phi_1(z), \phi_2(z), \phi_3(z), \ldots$, and natural frequencies $\omega_1, \omega_2, \omega_3, \ldots$. The principal modes relate to the vibration of the shaft as a beam, without rotation ($\Omega = 0$) and in the absence of damping ($b = 0 = h$). The modal solution depends upon the independence of the principal modes.

The forced whirl due to unbalance, ignoring any transient response associated with the complementary function of Eq. (8) has been shown to have the form,[1]

$$\xi = \sum_{r=1}^{\infty} \Omega^2 \bar{a}_r N_r(\Omega)\phi_r(z)e^{i(\Omega t - \zeta_r)} \tag{9}$$

where

$$N_r(\Omega) = [(\omega_r^2 - \Omega^2)^2 + 4\mu_r^2\omega_r^2\Omega^2]^{-1/2} \tag{10}$$

$$\zeta_r = \tan^{-1}[2\mu_r\omega_r\Omega/(\omega_r^2 - \Omega^2)] \tag{11}$$

$$\bar{a}_r = \frac{1}{Z}\int_0^{\ell} \rho A[\phi_r(z)]^2 \bar{a}(z)\,dz \tag{12}$$

Thus $\bar{a}_r$ represents the component of unbalance in the rth modes of the shaft.

In a similar way the forced whirl due to an initial bow can be reproduced with a slightly modified notation from Ref. 2 in the form

$$\xi = \sum_{r=1}^{\infty} \bar{e}_r \omega_r^2 N_r(\Omega)\phi_r(z)e^{i(\Omega t - \zeta_r)} \tag{13}$$

where

$$\omega_r^2 \bar{e}_r = \frac{1}{Z}\int_0^{\ell} \phi_r(z)\frac{d^2}{dz^2}\left[EI\frac{d^2\eta_0}{dz^2}\right]dz \tag{14}$$

It should be noted that, due to the orthogonality of the principal modes, Eq. (14) effectively implies that $\bar{e}_r$ is the component of the initial bow in the rth modes, such that

$$\eta_0 = \sum_{r=1}^{\infty} \bar{e}_r \phi_r(z) \tag{15}$$

Similarly, in most cases, the conventional unbalance can be expanded in a modal series

$$\bar{a}(z) = \sum_{r=1}^{\infty} \bar{a}_r \phi_r(z) \tag{16}$$

A comparison of Eqs. (9) and (13) shows that there is a marked similarity between the forced, steady whirl due to unbalance and that due to an initial bow. There is, however, one essential difference which is an important consideration in balancing, namely, that the unbalance effect of each term in series (9) is a multiple of Ω^2 and so varies with shaft speed, whereas the corresponding terms in series (13) are multiples of ω_r^2 ($r = 1,2,3,\ldots$) and so do not vary with shaft speed Ω. That is, both whirls undergo resonances due to harmonic forcing at a frequency $\Omega = \omega_r$ ($r = 1,2,3,\ldots$), but, in the case of mass unbalance, the amplitude of the forcing varies with Ω^2, whereas the forcing effect of an initial bow is independent of shaft speed Ω.

Although both forms of forced whirl demonstrate resonance behavior at the critical speeds, which correspond to $\omega_1, \omega_2, \omega_3, \ldots$, the character of the two whirls differ at very low and very high speeds. Thus, for a shaft with mass unbalance,

$$\lim_{\Omega \to 0} \xi = 0 \qquad \lim_{\Omega \to \infty} \xi = -\bar{a}(z)e^{i\Omega t} \tag{17}$$

whereas for a shaft with an initial bow,

$$\lim_{\Omega \to 0} \xi = \eta_0 e^{i\Omega t} = \xi_0 \qquad \lim_{\Omega \to \infty} \xi = 0 \tag{18}$$

Of course, as is quite likely, if a shaft suffers from both mass unbalance and initial bow, the forced whirl is a combination of Eqs. (9) and (13). That is,

$$\xi = \sum_{r=1}^{\infty} (\bar{a}_r\Omega^2 + \bar{e}_r\omega_r^2)N_r(\Omega)\phi_r(z)e^{i(\Omega t - \zeta_r)} \tag{19}$$

Balancing

Shafts with Mass Unbalance

The difference in the character of the forced whirls that have just been outlined are not great, but they are of additional significance in the context of balancing. The matter has been discussed in detail elsewhere.[2-6] Essentially, balancing is performed by attaching a correction mass distribution to the shaft, which produces an additional mass center eccentricity $\bar{b}(z)$ at each cross section with modal components

$$\bar{b}_r = \frac{1}{Z}\int_0^{\ell} \rho A \bar{b}(z)\phi_r(z)\,dz \tag{20}$$

In conventional balancing for mass unbalance, therefore, the correction masses are chosen so that

$$\bar{b}_r + \bar{a}_r = 0 \qquad (r = 1,2,3,\ldots,n) \tag{21}$$

for a finite series of modes which will normally ensure satisfactory operation of the shaft at all speeds up to a maximum value in the vicinity of the nth critical speed ω_n. No attempt is made to annul the unbalance completely, as such correction is not necessary and, in most cases, is not possible. The residual whirl after balancing has the form

$$\xi' = \sum_{r=n+1}^{\infty} (\bar{a}_r + \bar{b}_r)\Omega^2 N_r(\Omega)\phi_r(z)e^{i(\Omega t - \zeta_r)} \tag{22}$$

so that only the high, "nonresonant" modes remain to some extent uncorrected, although the net balance in these modes is modified by the balancing process. Note, in particular, that requirement (21) can be satisfied for all shaft speeds Ω.

Shafts with an Initial Bow

If a similar procedure is adopted for a shaft with an initial bow, but with no mass unbalance, the whirl resulting from the attachment of a correction mass distribution giving a mass center eccentricity $\bar{b}(z)$ can be written as

$$\xi' = \sum_{r=1}^{\infty} (\bar{e}_r\omega_r^2 + \bar{b}_r\Omega^2)N_r(\Omega)\phi_r(z)e^{i(\Omega t - \zeta_r)} \tag{23}$$

In these circumstances to annul the whirl in the rth mode requires a correction $\bar{b}(z)$ for which

$$\bar{e}_r\omega_r^2 + \bar{b}_r\Omega^2 = 0 \tag{24}$$

which can only be satisfied for one shaft speed Ω. In some respects, therefore, the "best" balance is achieved by ensuring that condition (24) is satisfied at the rth critical speed ω_r. That is the speed at which the whirl in the rth mode is likely to be large and important in practice due to modal resonance. This, indeed, was the policy advocated previously,[2,3] so that $\bar{b}(z)$ is selected, such that

$$\bar{e}_r + \bar{b}_r = 0 \tag{25}$$

for all modes, $r = 1,2,...,n$, that need to be corrected.

If the first n modes are balanced in this way, the residual whirl has the form

$$\xi' = \sum_{r=1}^{\infty} \bar{e}_r(\omega_r^2 - \Omega^2)N_r(\Omega)\phi_r(z)e^{i(\Omega t - \zeta_r)} + \sum_{r=n+1}^{\infty} (\bar{e}_r\omega_r^2 + \bar{b}_r\Omega^2)N_r(\Omega)\phi_r(z)e^{i(\Omega t - \zeta_r)} \tag{26}$$

Thus, there is a residual whirl in the higher, "uncorrected" modes ($r \geq n+1$) that is somewhat similar to that for a shaft with mass unbalance initially, except that the speed dependence of the residual forcing terms is not quite so simple. By contrast, however, there is still a whirl in the "balanced" lower modes ($r = 1,2,..,n$) for all speeds, apart from the corresponding critical speeds, although the amplitude of whirl in each balanced mode does not exceed the magnitude of $\bar{e}_r\phi_r(z)$. The variation in the amplitude of the whirl in the rth mode after balancing is illustrated by the curve labeled $\gamma_r = 1.0$ in Fig. 2, which is reproduced from Ref. 2.

The differences in the two residual whirls are manifested by their character at very low and very high speeds. In the case of a corrected shaft with mass unbalance initially, inspection of Eq. (22) shows that,

$$\lim_{\Omega\to 0} \xi' = 0 \qquad \lim_{\Omega\to\infty} \xi' = \sum_{r=n+1}^{\infty} (\bar{a}_r + \bar{b}_r)\phi_r(z)e^{i\Omega t} \tag{27}$$

whereas for the corresponding shaft with an initial bow Eq. (26) yields,

$$\lim_{\Omega\to 0} \xi' = \sum_{r=1} \bar{e}_r\phi_r(z)e^{i\Omega t} = \xi_0 \tag{28a}$$

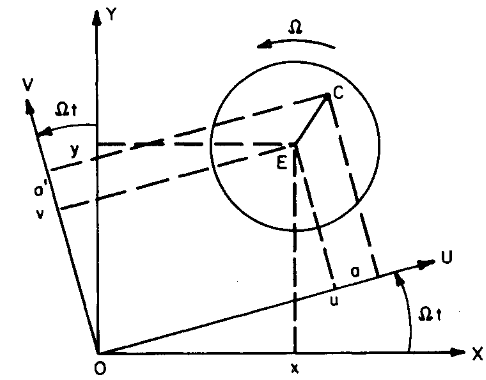

Fig. 1 Coordinate systems for a rotating shaft.

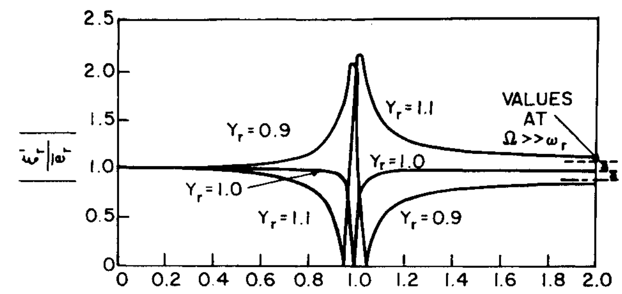

Fig. 2 Amplitude-speed curves in rth mode for a shaft having an initial bow: $\gamma_r = 0.9$, balance correction 10% low; $\gamma_r = 1.0$, perfect balance; $\gamma_r = 1.1$, balance correction 10% high; and $\xi' = \sum_{r=1}^{\infty} \xi_r'\phi_r(z)$.

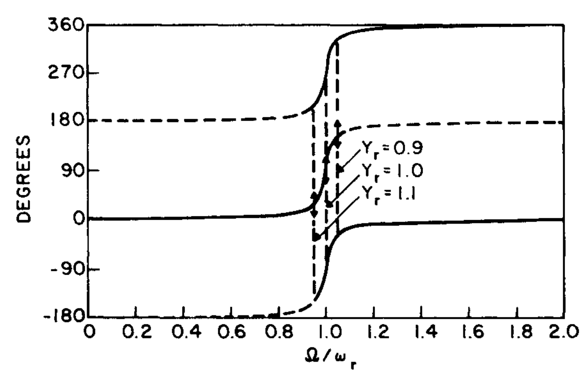

Fig. 3 Phase-speed curves in rth mode for a shaft having an initial bow: $\gamma_r = 0.9$, balance correction 10% low; $\gamma_r = 1.0$, perfect balance; and $\gamma_r = 1.1$, balance correction 10% high.

$$\begin{aligned}\lim_{\Omega\to\infty} \xi' &= \sum_{r=1}^{n} \bar{e}_r\phi_r(z)e^{i\Omega t} - \sum_{r=n+1}^{\infty} \bar{b}_r\phi_r(z)e^{i\Omega t} \\ &= \sum_{r=1}^{\infty} \bar{e}_r\phi_r(z)e^{i\Omega t} - \sum_{r=n+1}^{\infty} (\bar{e}_r + \bar{b}_r)\phi_r(z)e^{i\Omega t} \\ &= \xi_0 - \sum_{r=n+1}^{\infty} (\bar{e}_r + \bar{b}_r)\phi_r(z)e^{i\Omega t}\end{aligned} \tag{28b}$$

The shaft with an initial bow ξ_0 still demonstrates this effect after balancing, plus residual whirls in the uncorrected higher modes. By contrast, the latter are the only remaining displacements for a shaft with mass unbalance initially, which can be seen by comparing the limiting expressions of Eqs. (27) and (28).

The residual forcing term in the rth balancing mode $\bar{e}_r(\omega_r^2 - \Omega^2)$, moreover, modifies the variation in the phase of the

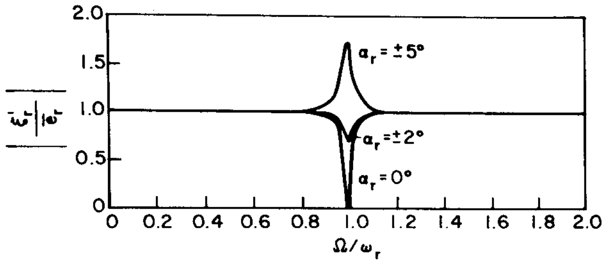

Fig. 4 Amplitude-speed curves in rth mode for a shaft having an initial bow: $\alpha_r = 0$ deg, correct angular location of balance correction; $\alpha_r = \pm 2$ deg, error in angular location of balance correction; $\alpha_r = \pm 5$ deg, error in angular location of balance correction; and $\xi' = \sum_{r=1}^{\infty} \xi_r' \phi_r(z)$.

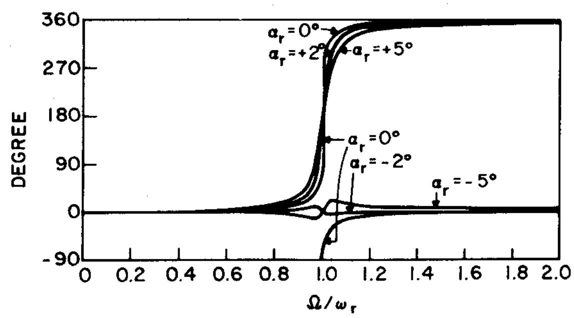

Fig. 5 Phase-speed curves in rth mode for a shaft having an initial bow: $\alpha_r = 0$ deg, correct angular location of balance correction; $\alpha_r = \pm 2$ deg, error in angular location of balance correction; and $\alpha_r = \pm 5$ deg, error in angular location of balance correction.

whirl with speed, as this term itself changes in phase by 180 deg as the shaft speed is increased from below to above the critical speed ω_r. The phase variations with shaft speed are illustrated by the curves for $\gamma_r = 1.0$ in Fig. 3 which is reproduced from Ref. 2.

The technique just given is still applicable, if a shaft has mass unbalance and an initial bow. The whirl in the rth mode, represented by the rth term in series (19) is balanced by selecting a correction $\bar{b}(z)$ for which

$$\bar{a}_r \Omega^2 + \bar{e}_r \omega_r^2 + \bar{b}_r \Omega^2 = 0 \tag{29}$$

for $\Omega = \omega_r$; that is,

$$\bar{b}_r = -(\bar{a}_r + \bar{e}_r) \tag{30}$$

Consequently, if the first n modes are corrected in this way, the residual whirl can be expressed in the form

$$\xi' = \sum_{r=1}^{n} \bar{e}_r (\omega_r^2 - \Omega^2) N_r(\Omega) \phi_r(z) e^{i(\Omega t - \zeta_r)} + \sum_{r=n+1}^{\infty} (\bar{e}_r \omega_r^2 + \bar{a}_r \Omega^2 + \bar{b}_r \Omega^2) N_r(\Omega) \phi_r(z) e^{i(\Omega t - \zeta_r)} \tag{31}$$

The residual whirl, therefore, is very similar to that for a "balanced" shaft which initially only had an initial bow and no mass unbalance [compare Eqs. (26) and (31)].

The presence of an initial bend has been seen to introduce no real difficulty in principle, provided that the aim of the balancing procedure is modified, so that the whirl in a mode to be balanced is only annulled at the corresponding critical speed. The residual modal whirl will not exceed the modal component of the initial bow and is acceptable in general. In practice, however, balancing such shafts is not so simple, due to the variations in the residual forcing terms, in both amplitude and phase, with shaft speed. Small, unavoidable errors in balancing produce unexpected properties in the amplitude and phase of the residual whirl, so that if the effects of balancing are assessed by means of amplitude and phase observations, it is very easy to misinterpret the effects of a correction mass distribution.

Figures 2 and 3 illustrate the results of balancing a shaft with initial bow in the rth mode by means of attaching a balance mass distribution at the correct angular location around the shaft, but whose magnitude is given as a multiple, $\gamma_r = 0.9$, 1.0, or 1.1, of the correct value. Thus, the curves for $\gamma_r = 1.0$ represent a "perfect" balance, while those for $\gamma_r = 0.9$ and 1.1 illustrate, respectively, the effect of under- and overbalancing by 10%.

Somewhat similar characteristics are shown in Figs. 4 and 5, which are also reproduced from Ref. 2. Here, the balancing distribution is assumed to have the correct magnitude, but its angular location around the shaft is in error by an angle, $\alpha_r = 0, \pm 2, \pm 5$ deg, relative to the correct location. A perfect balance is represented by the curves for $\alpha_r = 0$ deg, which are identical to those for $\gamma_r = 1.0$ in Figs. 2 and 3.

If behavior of the form illustrated in Figs. 2-5 is discovered while balancing a shaft, this is a positive indication that the shaft has some initial bow and that extra care must be taken in assessing the effects of the corrections. Indeed, practical, industrial engineers often have reported behavior of this kind and were reassured by an explanation in terms of initial bend.

Alternative Procedure for Shafts with Initial Bow

It is often suggested that, if a shaft suffers from an initial bow, then the initial bow, determined by measuring the low-speed runout, can be subtracted from the whirl measured at speed and that just the mass unbalance whirl is left. Inspection of Eq. (19) shows, however, that the effects of the initial bow cannot be simply subtracted out in this way. For the purposes of the present discussion, it is convenient to rewrite Eq. (19) as

$$\xi = \sum_{r=1}^{\infty} (\bar{a}_r \Omega^2 + \bar{e}_r \omega_r^2) N_r^*(\Omega) \phi_r(z) e^{i\Omega t} \tag{32}$$

where

$$N_r^*(\Omega) = [(\omega_r^2 - \Omega^2) + 2i\mu_r \omega_r \Omega]^{-1} \tag{33}$$

so that $N_r^*(\Omega)$ is a complex factor whose magnitude is $N_r(\Omega)$, defined in Eq. (10). Consequently, subtraction of the low-speed runout, or initial bow ξ_0, gives

$$\xi - \xi_0 = \sum_{r=1}^{\infty} [\bar{a}_r \Omega^2 + \bar{e}_r (\Omega^2 - 2i\mu_r \omega_r \Omega)] N_r^*(\Omega) \phi_r(z) e^{i\Omega t} \tag{34}$$

Equation (34) shows that the net whirl $\xi - \xi_0$ contains some components due to mass unbalance and others due to the initial bow.

A comparison of Eqs. (32) and (34), however, indicates that there may well be advantages in attempting to balance the net whirl $\xi - \xi_0$ rather than the total whirl ξ, provided that the balancing engineer understands clearly what is being attempted. For example, to balance the component of the net whirl in the rth mode, the correction masses would be chosen so that the associated rth mode component $\bar{b}_r$ is

$$\bar{b}_r = -\bar{a}_r - \bar{e}_r (1 - 2i\mu_r) \tag{35}$$

Attachment of correction (35) reduces the forcing term in the rth mode in Eq. (34) from

$$\bar{a}_r \Omega^2 + \bar{e}_r (\Omega^2 - 2i\mu_r \omega_r \Omega) \tag{36a}$$

to

$$\bar{e}_r 2i\mu_r\Omega(\Omega-\omega_r) \tag{36b}$$

The residual forcing term (36b) is much smaller than the initial forcing function (36a), as the modal damping coefficient μ_r is small. In addition, the multiplying factor $(\Omega-\omega_r)$ is small at shaft speeds close to the critical speed ω_r. It will be remembered that the unbalance whirl in the rth mode is normally only important at shaft speeds close to ω_r.

Balancing the net whirl $\xi-\xi_0$ in this way, therefore, ensures that the residual total whirl ξ' is approximately equal to the initial bow ξ_0. By contrast, correction of the total whirl ξ results in a residual whirl which is zero at the critical speeds of the balanced modes, but approaches the initial bow ξ_0 at off-resonant speeds. As the initial bow, like mass unbalance, is normally a small defect which is only important due to magnification at resonance, both balancing procedures are likely to produce satisfactory operation of the shaft. In principle, therefore, correction of the total whirl ξ produces a somewhat smaller residual whirl, but, in practice, the difference will not be significant. Numerical illustrations of some of these aspects may be found in Refs. 2, 4, and 5.

Indeed, it may well be easier to balance the net whirl, rather than the total whirl. The forcing term of the net whirl, $\xi-\xi_0$, in the rth mode is proportional to

$$(\bar{a}_r+\bar{e}_r)\Omega^2-2i\mu_r\omega_r\Omega\bar{e}_r \tag{37}$$

from Eq. (36a), whereas the corresponding term for the total whirl is

$$\bar{a}_r\Omega^2+\bar{e}_r\omega_r^2 \tag{38}$$

In the case of the total whirl, therefore, the forcing term, represented by Eq. (38), varies significantly in both magnitude and phase with shaft speed Ω. This variation can make the determination of the forcing term at the critical speed $\Omega=\omega_r$ difficult, although this determination is essential for balancing the total whirl. In addition, it is the interaction of the speed dependent $\bar{a}_r\Omega^2$ term and constant $\bar{e}_r\omega_r^2$ factor which leads to the unusual and potentially misleading amplitude and phase relationships of Figs. 2-5, which can further add to the complications of balancing.

By contrast, the nature of the forcing in Eq. (37) for the net whirl is very similar to $\bar{a}_r\Omega^2$ for mass unbalance in Eq. (9). As the $2\mu_r\omega_r\Omega\bar{e}_r$ term is likely to be very small relative to the $(\bar{a}_r+\bar{e}_r)\Omega^2$ component, due to the damping coefficient μ_r being small, the forcing term essentially varies in magnitude with Ω^2 but its phase is independent of shaft speed. The determination of the magnitude and phase of the net unbalance $\bar{a}_r+\bar{e}_r$ is relatively simple and can be achieved by direct application of conventional balancing techniques, using trial masses. Moreover, the variation of the amplitude and phase of the net whirl with speed will conform with the conventional resonance behavior of a vibrating system displaying a peak amplitude at the critical speed and a 180-deg change in phase as the shaft speed is increased through the critical speed.

The relative insignificance of the $2\mu_r\omega_r\Omega\bar{e}_r$ term in Eq. (37) can be illustrated by considering a shaft with an initial bow but no mass unbalance. The forcing term in Eq. (37) becomes

$$\bar{e}_r\Omega^2-2i\mu_r\omega_r\Omega\bar{e}_r=\bar{e}_r\Omega^2\,[1-2i\mu_r(\omega_r/\Omega)]$$

$$=\bar{e}_r\Omega^2\,[1+4\mu_r^2(\omega_r/\Omega)^2]^{1/2}e^{-i\theta_r} \tag{39}$$

where

$$\tan\theta_r=2\mu_r\omega_r\Omega \tag{40}$$

Values for θ_r and the multiplying factor $[1+4\mu_r^2(\omega_r/\Omega)^2]^{1/2}$ are given in Tables 1 and 2 for $\mu_r=0.025$ and $\mu_r=0.05$, respectively, which are typical values suggested for the modal damping coefficient by the results in Ref. 3, especially $\mu_r=0.025$. Considering that balancing the rth mode is usually attempted at shaft speeds Ω between 90% and 100% of the critical speed ω_r, an "error" of about -3 deg is likely in the determination of the angular orientation of the unbalance $\bar{e}_r\Omega^2$ around the rotor, when $\mu_r=0.025$. In addition, the magnitude of $\bar{e}_r\Omega^2$ is likely to be overestimated by approximately 0.14%! The corresponding errors, when $\mu_r=0.05$, are -6 deg and 0.6%.

Values for the modal phase lag ζ_r and the modal magnification factor $\Omega^2N(\Omega)$ are also listed in Tables 1 and 2. Inspection of these tables shows that for shaft speeds Ω, such that $0.9<\Omega/\omega_r<1.0$, the aforementioned errors are probably smaller than the discrepancies in observed values for ζ_r and the magnitude of the modal component of the whirl due to slight variations in speed control, damping, and other practical considerations, which are likely in practical, industrial balancing.

Thus, in practice the net whirl $\xi-\xi_0$ can be balanced as if it was due to mass unbalance effects alone, although the forcing term may consist effectively of $(\bar{a}_r+\bar{e}_r)\Omega^2$, and the principal, difficult aspects of initial bend only arise if attention is directed at the total whirl ξ. Moreover, the minor inaccuracies which occur in determining the factor $(\bar{a}_r+\bar{e}_r)$ in this way at speeds $\Omega<\omega_r$ take the form of a small negative error in angular location $-\theta_r$ and a very slight overestimate in magnitude.

Experimental Results

Some of the effects of initial bow were observed during a series of balancing trials that have been fully reported elsewhere.[7-10] In fact, the shaft which was the subject of these trials was whirling mainly due to initial bow, rather than mass unbalance. For this reason the balancing was performed in terms of the net whirl $\xi-\xi_0$ and not the total whirl ξ, although this aspect was not stressed in the reports. Nevertheless, the results reported in Refs. 7-10 can be regarded as a vindication of the treatment of shafts with initial bow by balancing the net whirl.

The test shaft was tubular, of length 3.7 m, external diameter 7.62 cm, and wall thickness 3.175 mm. It was supported on two bearings and in a damper which, apart from stabilizing the shaft, heavily damped the whirl at the second critical speed. The whirl of the shaft was detected by means of proximity, vibration transducers, located at several axial positions along its length. In the test configuration the rig had a maximum speed of 10,000 rpm and the first four critical speeds were, approximately, $\omega_1=925$ rpm, $\omega_2=2500$ rpm, $\omega_3=3750$ rpm, and $\omega_4=7900$ rpm. The test shaft was, therefore, of good size and was not an unrealistically small laboratory model. The test rig is described in more detail in Refs. 7-10.

Figure 6 illustrates the variation of the amplitude of the total whirl of the shaft with speed at an early stage in the process of balancing the fourth mode. The characteristic behavior of a balanced bent shaft is demonstrated at the first and fourth critical speeds. The amplitude at the first critical speed is very low and, in fact, is zero at a speed just below ω_1. A small resonance is shown at the first critical speed, so that the shaft is clearly slightly overbalanced in its first mode. This may be seen by comparing Fig. 6 with the curve for $\gamma_r=1.1$ in Fig. 2. In a similar way, the amplitude of whirl decreases to zero at approximately 6875 rpm before rising toward a resonance at the fourth critical speed. At that stage in the balancing process the shaft was also overbalanced in its fourth mode. Indeed, the magnitude of $(7900/6875)^2=1.32$ suggests that the correction for the fourth mode was some 30% too high, but this estimate could not be confirmed, as the shaft hit one of the transducer probe shields, so that the balancing trial had to be partially repeated.

A similar amplitude-speed curve is shown in Fig. 7 which relates to a subsequent stage in the balancing trial. Once again the whirl amplitude in the first mode is very well controlled. A very small resonance "peak" is demonstrated at the critical speed and this time the amplitude decreases almost to zero at a speed just above the first critical speed, before increasing once more. The shaft was, thus, marginally undercorrected in its first mode. In both Figs. 6 and 7 the residual, nonzero amplitude below the first critical speed demonstrates the presence of an initial bow. The initial bow was also detected by measurements of the low-speed runout ξ_0 at locations along the shaft at 350 rpm. In addition, a phase change of some 350 deg was observed during the transition through the first critical speed during the tests relating to Figs. 6 and 7.

In Figs. 6 and 7 small resonance peaks can be seen at the second critical speed, but this mode did not require balancing due to the effect of the damper referred to earlier. The effects of bow were not detected in balancing the third mode of the shaft, which suggests that the form of the initial bow and the shape $\phi_3(z)$ of the third mode resulted in an insignificant value for e_3, the third mode component of initial bend [see Eqs. (14) and (15)]. The resonance at the third critical speed must be largely due to mass unbalance.

The effects of an initial bow in the first mode are also demonstrated in Figs. 8a and 8b. Figure 8a shows the amplitude-speed variation near the first critical speed for the total whirl ξ and the net whirl $\xi - \xi_0$, while the associated phase-speed relationships are plotted in Fig. 8b. The amplitude of the net whirl shows a conventional resonance peak, whereas the amplitude of the total whirl decreases approximately to zero at 860 rpm before rising to a resonance at the first critical speed. Moreover, the phase of the total whirl changes by 330 deg between 800 and 970 rpm, whereas the phase of the net whirl only changes by 167 deg.

Finally, Figs. 9a and 9b depict the response of the shaft to a discrete mass unbalance alone. The response was calculated by measuring the whirl of the shaft at speeds from 750 to 950 rpm and repeating the measurements after the attachment of a single mass, 0.93 g, at a convenient location along the shaft. Vector subtraction of the two sets of vibration readings yielded the response to the mass alone. The amplitude-speed and phase-speed curves derived in this way give the response of mass unbalance alone; any effects of the initial bend having

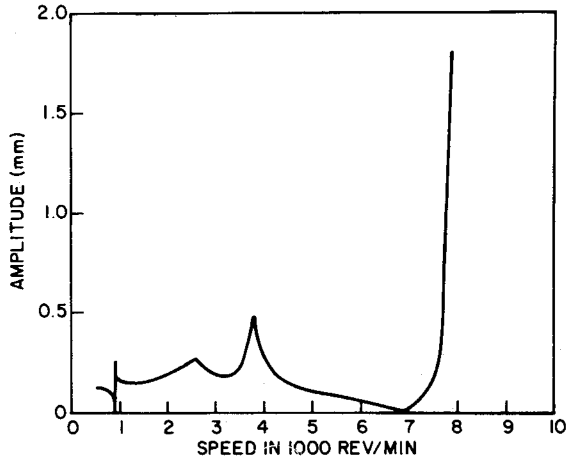

Fig. 6 Whirl of shaft during balancing.

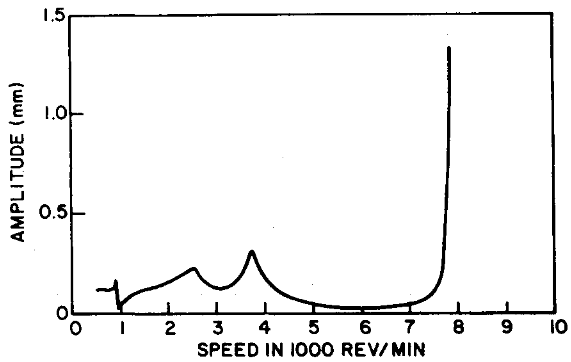

Fig. 7 Whirl of shaft at a later stage during balancing.

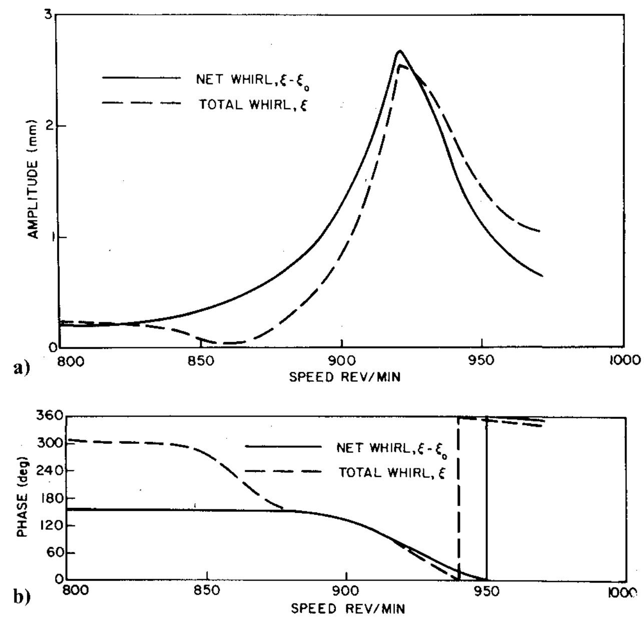

Fig. 8 Whirl of shaft due to initial bow. a) Amplitude. b) Phase.

Table 1 Errors due to balancing net whirl with $\mu_r = 0.025$

Ω/ω_r	0.7	0.8	0.9	1.0	1.1	1.2	1.3
θ_r, deg	4.1	3.6	3.2	2.9	2.6	2.4	2.2
ζ_r, deg	3.9	6.3	13.3	90	165.3	172.2	174.6
$[1+4\mu_r^2(\omega_r/\Omega)^2]^{1/2}$	1.0025	1.0020	1.0015	1.0012	1.0010	1.0009	1.0007
$\Omega^2 N_r(\Omega)$	0.96	1.77	4.15	20	5.54	3.14	2.44

Table 2 Errors due to balancing net whirl with $\mu_r = 0.05$

Ω/ω_r	0.7	0.8	0.9	1.0	1.1	1.2	1.3
θ_r, deg	8.1	7.1	6.3	5.7	5.2	4.8	4.4
ζ_r, deg	7.8	12.5	25.4	90	152.4	164.7	169.3
$[1+4\mu_r^2(\omega_r/\Omega)^2]^{1/2}$	1.0102	1.0078	1.0062	1.0050	1.0041	1.0035	1.0030
$\Omega^2 N_r(\Omega)$	0.95	1.74	3.76	10	5.10	3.16	2.41

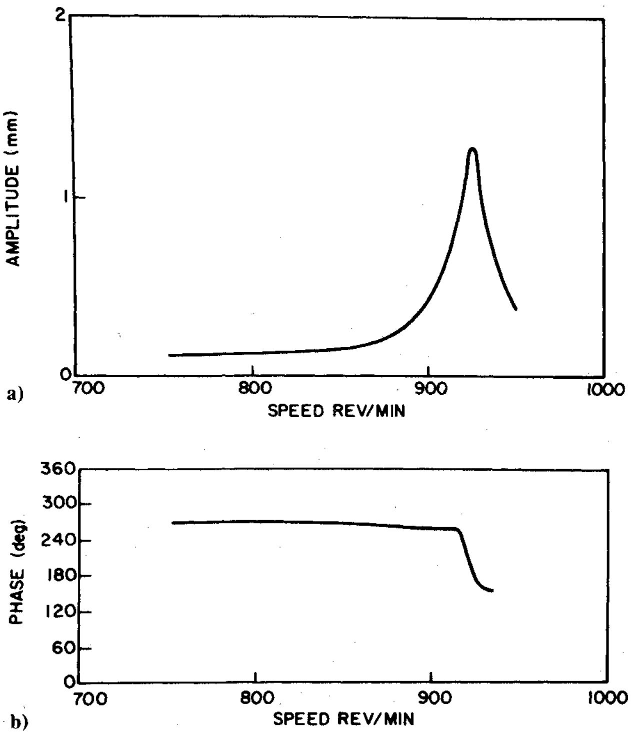

Fig. 9 Whirl of shaft due to a discrete mass unbalance. a) Amplitude. b) Phase.

been subtracted out. The curves in Figs. 9a and 9b conform to the pattern expected for conventional, mass unbalance whirl. They demonstrate a resonance peak at the first critical speed and a phase change of 120 deg. The full 180-deg phase change is not displayed, as the curves are only plotted up to a speed of 950 rpm, that is, some 25 rpm above the critical speed.

Conclusions

There is a very real difference between the synchronous response of a bent shaft as opposed to one with a pure mass unbalance. It has been shown that balancing the net whirl of a bent shaft is effective, if not exactly correct. This has been demonstrated analytically and verified by experimental results which demonstrated effective balancing of a bent shaft through several flexural critical speeds based on net whirl measurements.

Acknowledgment

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